LISTA 1, zad 10

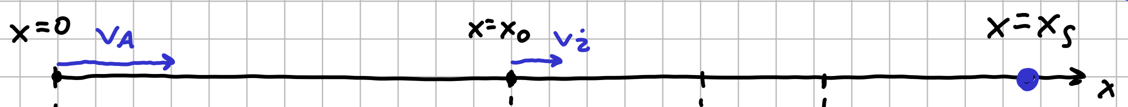
paradoks Zenona

(1) $x^{A}(t) = v_A t$

(2) $x^{\dot{z}}(t) = x_0 + v_{\dot{z}} t$

odcinki drogi i czasu

miejsce spotkania (s) A i Ż



t_s - czas spotkania

$$\left.\begin{array}{l} \Delta x_1^A > \Delta x_2^A > \Delta x_3^A > \dots \\ \Delta t_1 > \Delta t_2 > \Delta t_3 > \dots \end{array}\right\} A$$

$$\left.\begin{array}{l} \Delta x_1^{\dot{z}} > \Delta x_2^{\dot{z}} > \dots \\ \Delta t_1 > \Delta t_2 > \dots \end{array}\right\} \dot{Z}$$

(3) $\Delta x_j^A = v_A \Delta t_j$

(4) $\Delta x_j^{\dot{z}} = v_{\dot{z}} \Delta t_j$

(5) $\Delta x_j^A = \begin{cases} x_0, & j=1 \\ \Delta x_{j-1}^{\dot{z}}, & j \geq 2 \end{cases}$

↓ łącząc (3, 4, 5)

(5') $v_A \Delta t_j = \begin{cases} x_0, & j=1 \\ v_{\dot{z}} \Delta t_{j-1}, & j \geq 2 \end{cases}$

czyli

(5'') $\Delta t_j = \begin{cases} \frac{x_0}{v_A}, & j=1 \\ \frac{v_{\dot{z}}}{v_A} \Delta t_{j-1}, & j \geq 2 \end{cases}$

Podsumowując, dla każdego $j \geq 1$, $\boxed{\Delta t_j = \frac{x_0}{v_A}\left(\frac{v_{\dot{z}}}{v_A}\right)^{j-1}}$ (6)

Wiemy, że

$$t_s = \sum_{j=1}^{\infty} \Delta t_j \overset{l=j-1}{=} \frac{x_0}{v_A} \sum_{l=0}^{\infty} \left(\frac{v_{\dot{z}}}{v_A}\right)^{l} \overset{\text{suma szeregu geometrycznego, } v_{\dot{z}} < v_A}{=} \frac{x_0}{v_A} \frac{1}{1 - \frac{v_{\dot{z}}}{v_A}} = \frac{x_0}{v_A - v_{\dot{z}}} < \infty$$

□

Zenon nie przewidział, że suma ∞ elementów może być skończona.

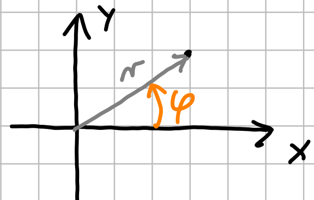
WSTĘP DO LISTY 2

Wsp. kartezjańskie (x, y)

$$\vec{r} = [x, y] = \hat{i}\, x + \hat{j}\, y \qquad (0)$$

wersory $\hat{i}, \hat{j}$

Wsp. biegunowe (r, φ)



$$\begin{cases} x = r\cos\varphi & (1a) \\ y = r\sin\varphi & (1b) \end{cases}$$

wersory $\hat{r}, \hat{\varphi}$

Współczynniki Lamego H_r, H_φ

$$\rightarrow \frac{\partial \vec{r}}{\partial r} = \frac{\partial [r\cos\varphi, r\sin\varphi]}{\partial r} = [\cos\varphi, \sin\varphi] = \hat{i}\cos\varphi + \hat{j}\sin\varphi \qquad (2a)$$

$$H_r = \left|\frac{\partial \vec{r}}{\partial r}\right| \overset{(2a)}{=} \sqrt{\cos^2\varphi + \sin^2\varphi} = 1 \qquad (3a)$$

$$\rightarrow \frac{\partial \vec{r}}{\partial \varphi} = \frac{\partial [r\cos\varphi, r\sin\varphi]}{\partial \varphi} = [-r\sin\varphi, r\cos\varphi] =$$

$$= -\hat{i}\, r\sin\varphi + \hat{j}\, r\cos\varphi \qquad (2b)$$

$$H_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| \overset{(2b)}{=} \sqrt{(-r\sin\varphi)^2 + (r\cos\varphi)^2} = r \qquad (3b)$$

Wersory

$$\rightarrow \hat{r} = \frac{1}{H_r}\frac{\partial \vec{r}}{\partial r} \overset{(2a, 3a)}{=} \hat{i}\cos\varphi + \hat{j}\sin\varphi \qquad (4a)$$

$$\rightarrow \hat{\varphi} = \frac{1}{H_\varphi}\frac{\partial \vec{r}}{\partial \varphi} \overset{(2b, 3b)}{=} -\hat{i}\sin\varphi + \hat{j}\cos\varphi \qquad (4b)$$

Z (4) mamy, że $\hat{r} \cdot \hat{\varphi} = 0$, zatem $\hat{r} \perp \hat{\varphi}$

Z (0, 1, 4a) otrzymujemy $\boxed{\vec{r} = r\,\hat{r}}$ (5)

Pochodne po czasie z wersorów

$$\rightarrow \dot{\hat{r}} \overset{(4a)}{=} \hat{i}(\cos\varphi)' + \hat{j}(\sin\varphi)' = -\hat{i}\,\dot{\varphi}\sin\varphi + \hat{j}\,\dot{\varphi}\cos\varphi \overset{(4b)}{=} \dot{\varphi}\,\hat{\varphi} \qquad (6a)$$

$$\rightarrow \dot{\hat{\varphi}} \overset{(4b)}{=} -\hat{i}(\sin\varphi)' + \hat{j}(\cos\varphi)' = -\hat{i}\,\dot{\varphi}\cos\varphi - \hat{j}\,\dot{\varphi}\sin\varphi \overset{(4a)}{=} -\dot{\varphi}\,\hat{r} \qquad (6b)$$

Komentarz do zad. 1c, lista 2:

Wyprowadziliśmy, że

$$\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (\underbrace{2\dot{r}\dot{\varphi}}_{\text{dł. wektora przyspieszenia Coriolisa}, \vec{a}_c} + \overbrace{r\ddot{\varphi}}^{\text{przysp. kątowe}})\hat{\varphi}$$

$\vec{a}_c$ – pozorne przyspieszenie występujące w obracającym się układzie odniesienia

np. człowiek idący od środka do krawędzi obracającej się karuzeli czuje, że znosi go na bok

np. wahadło Foucaulta

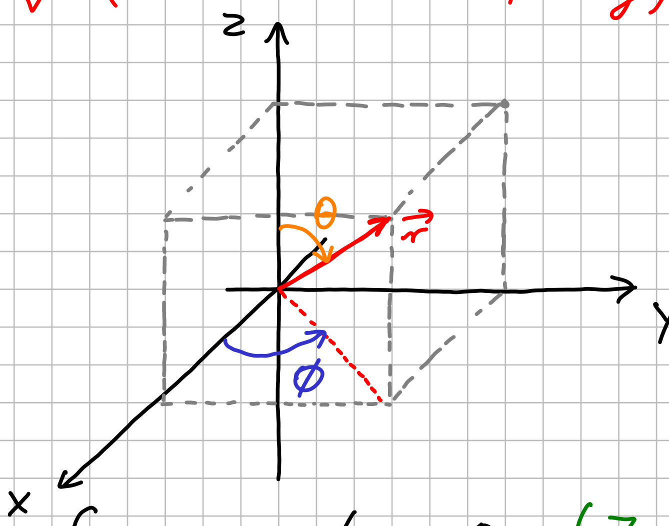
A teraz – 3D

Wsp. kartezjańskie

$$\vec{r} = [x, y, z] = \hat{i}x + \hat{j}y + \hat{k}z \qquad (0')$$

Wsp. sferyczne

⚠ (ukł. matematyczny)



$$\begin{cases} x = r\cos\phi\sin\theta & (7a) \\ y = r\sin\phi\sin\theta & (7b) \\ z = r\cos\theta & (7c) \end{cases}$$

Na ćwiczeniach wyprowadziliśmy, że

wsp. Lamego to:

$$\begin{cases} H_r = 1 & (8a) \\ H_\theta = r & (8b) \\ H_\phi = r\sin\theta & (8c) \end{cases}$$

wersory to:

$$\begin{cases} \hat{r} = [\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta] & (9a) \\ \hat{\theta} = [\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta] & (9b) \\ \hat{\phi} = [-\sin\phi, \cos\phi, 0] & (9c) \end{cases}$$

Zatem, z (0', 7, 9), tak jak dla wsp. biegunowych

$$\boxed{\vec{r} = r\,\hat{r}} \qquad (10)$$

Pochodne wersorów po czasie:

$$\dot{\hat{r}} \overset{(9a)}{=} \hat{i}\,(\cos\phi\sin\theta)' + \hat{j}\,(\sin\phi\sin\theta)' + \hat{k}\,(\cos\theta)' =$$
$$= \hat{i}\,(-\dot{\phi}\sin\phi\sin\theta + \dot{\theta}\cos\phi\cos\theta)$$
$$+ \hat{j}\,(\dot{\phi}\cos\phi\sin\theta + \dot{\theta}\sin\phi\cos\theta) - \hat{k}\,\dot{\theta}\sin\theta =$$
$$= \dot{\phi}\sin\theta\,(-\hat{i}\sin\phi + \hat{j}\cos\phi) + \dot{\theta}\,(\hat{i}\cos\phi\cos\theta + \hat{j}\sin\phi\cos\theta - \hat{k}\sin\theta) =$$
$$\overset{(9bc)}{=} (\dot{\phi}\sin\theta)\,\hat{\phi} + \dot{\theta}\,\hat{\theta} = \dot{\hat{r}} \qquad (11a)$$

$$\dot{\hat{\theta}} \overset{(9b)}{=} \hat{i}\,(\cos\phi\cos\theta)' + \hat{j}\,(\sin\phi\cos\theta)' - \hat{k}\,(\sin\theta)' =$$
$$= \hat{i}\,(-\dot{\phi}\sin\phi\cos\theta - \dot{\theta}\cos\phi\sin\theta)$$
$$+ \hat{j}\,(\dot{\phi}\cos\phi\cos\theta - \dot{\theta}\sin\phi\sin\theta) - \hat{k}\,\dot{\theta}\cos\theta =$$
$$= -\dot{\theta}\,(\hat{i}\cos\phi\sin\theta + \hat{j}\sin\phi\sin\theta + \hat{k}\cos\theta)$$
$$+ \dot{\phi}\cos\theta\,(-\hat{i}\sin\phi + \hat{j}\cos\phi) =$$
$$\overset{(9ac)}{=} -\dot{\theta}\,\hat{r} + (\dot{\phi}\cos\theta)\,\hat{\phi} = \dot{\hat{\theta}} \qquad (11c)$$

$$\dot{\hat{\phi}} \overset{(9c)}{=} -\hat{i}\,(\sin\phi)' + \hat{j}\,(\cos\phi)' = -\hat{i}\,\dot{\phi}\cos\phi - \hat{j}\,\dot{\phi}\sin\phi =$$
$$= -\hat{i}\,\dot{\phi}\cos\phi\,(\sin^2\theta + \cos^2\theta) - \hat{j}\,\dot{\phi}\sin\phi\,(\sin^2\theta + \cos^2\theta)$$
$$- \hat{k}\,\dot{\phi}\sin\theta\cos\theta + \hat{k}\,\dot{\phi}\cos\theta\sin\theta =$$
$$= -\dot{\phi}\sin\theta\,(\hat{i}\cos\phi\sin\theta + \hat{j}\sin\phi\sin\theta + \hat{k}\cos\theta)$$
$$-\dot{\phi}\cos\theta\,(\hat{i}\cos\phi\cos\theta + \hat{j}\sin\phi\cos\theta - \hat{k}\sin\theta) \overset{(9ab)}{=}$$
$$= -(\dot{\phi}\sin\theta)\,\hat{r} - (\dot{\phi}\cos\theta)\,\hat{\theta} = \dot{\hat{\phi}} \qquad (11b)$$