1. STAN KOHERENTNY $|\alpha\rangle$

a) Stan własny operatora anihilacji:

$$\boxed{\hat{a}\,|\alpha\rangle = \underset{\downarrow}{\alpha}\,|\alpha\rangle} \quad (1)$$

⚠ wartość własna $\alpha \in \mathbb{C}$, bo $\hat{a}$ nie jest hermitowski

b) Baza stanów Focka:

→ pamiętamy, że $$|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}\,|0\rangle_{vac} \quad (2)$$

→ $$|\alpha\rangle = \hat{1}\,|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\underbrace{\langle n|\alpha\rangle}_{c_n} = \sum_{n=0}^{\infty} c_n |n\rangle = |\alpha\rangle \quad (3)$$

→ $$c_n = \langle n|\alpha\rangle \overset{(2)}{=} \frac{1}{\sqrt{n!}}\,{}_{vac}\langle 0|\,\hat{a}^n|\alpha\rangle \overset{(1)}{=} \frac{\alpha^n}{\sqrt{n!}}\,{}_{vac}\langle 0|\alpha\rangle \quad (4)$$

→ normowanie - chcemy, by $\langle\alpha|\alpha\rangle = 1$, stąd

$$1 = \langle\alpha|\alpha\rangle \overset{(3)}{=} \sum_{n,m=0}^{\infty} \langle n|c_n^* c_m|m\rangle = \sum_{n,m=0}^{\infty} c_n^* c_m \overbrace{\langle n|m\rangle}^{=\delta_{nm}} =$$

$$= \sum_{n=0}^{\infty} |c_n|^2 \overset{(4)}{=} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}\,|{}_{vac}\langle 0|\alpha\rangle|^2 = e^{|\alpha|^2}|{}_{vac}\langle 0|\alpha\rangle|^2$$

zatem $|{}_{vac}\langle 0|\alpha\rangle|^2 = e^{-|\alpha|^2}$, co daje

$${}_{vac}\langle 0|\alpha\rangle = e^{-\frac{|\alpha|^2}{2} + i\phi} \quad (5)$$

Podsumowując, z (3, 4, 5) mamy, że:

$$\boxed{|\alpha\rangle = e^{i\phi}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,e^{-\frac{|\alpha|^2}{2}}\,|n\rangle} \quad (6)$$

⚠ Jak wiadomo, stan kwantowy jest określony z dokładnością do fazy globalnej, więc można przyjąć, że $\phi = 0$.

c) Prawdopodobieństwo wystąpienia n fotonów w stanie koherentnym $|\alpha\rangle$.

$$p_n = |c_n|^2 \overset{(4,5)}{=} \frac{|\alpha|^{2n}}{n!}\,e^{-|\alpha|^2} \quad (7)$$

→ ⚠ rozkład Poissona o parametrze średniej $\lambda = |\alpha|^2$

2. OPERATORY KWADRATUR

$$\boxed{\hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^+\right) \quad (8a) \quad \Big| \quad \hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^+\right) \quad (8b)}$$

$$\to \langle \alpha | \hat{X}_1 | \alpha \rangle \overset{(8a)}{=} \frac{1}{2}\left(\langle \alpha | \hat{a} | \alpha \rangle + \langle \alpha | \hat{a}^+ | \alpha \rangle\right) \overset{(1)}{=} \frac{1}{2}\left(\alpha + \alpha^*\right) = Re(\alpha) \quad (9a)$$

$$\to \langle \alpha | \hat{X}_2 | \alpha \rangle \overset{(8b)}{=} \frac{1}{2i}\left(\langle \alpha | \hat{a} | \alpha \rangle - \langle \alpha | \hat{a}^+ | \alpha \rangle\right) \overset{(1)}{=} \frac{1}{2i}\left(\alpha - \alpha^*\right) = Im(\alpha) \quad (9b)$$

$$\to \hat{X}_1^2 = \frac{1}{4}\left(\hat{a} + \hat{a}^+\right)^2 = \frac{1}{4}\left(\hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) = \left\{ [\hat{a}, \hat{a}^+] = 1 \right\}$$

$$= \frac{1}{4}\left(\hat{a}^2 + \hat{a}^{+2} + 2\hat{a}^+\hat{a} + 1\right) \quad (10a)$$

$$\to \hat{X}_2^2 = -\frac{1}{4}\left(\hat{a} - \hat{a}^+\right)^2 = -\frac{1}{4}\left(\hat{a}^2 + \hat{a}^{+2} - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}\right) =$$

$$= -\frac{1}{4}\left(\hat{a}^2 + \hat{a}^{+2} - 2\hat{a}^+\hat{a} - 1\right) \quad (10b)$$

$$\to \langle \alpha | \hat{X}_1^2 | \alpha \rangle \overset{(10a)}{=} \frac{1}{4}\left(\langle \alpha | \hat{a}^2 | \alpha \rangle + \langle \alpha | \hat{a}^{+2} | \alpha \rangle + 2\langle \alpha | \hat{a}^+\hat{a} | \alpha \rangle + \langle \alpha | \alpha \rangle\right) =$$

$$= \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1\right) =$$

$$= \frac{1}{4}\left(\left(\alpha + \alpha^*\right)^2 + 1\right) = \left(Re\,\alpha\right)^2 + \frac{1}{4} \quad (11a)$$

$$\to \langle \alpha | \hat{X}_2^2 | \alpha \rangle \overset{(10b)}{=} -\frac{1}{4}\left(\langle \alpha | \hat{a}^2 | \alpha \rangle + \langle \alpha | \hat{a}^{+2} | \alpha \rangle - 2\langle \alpha | \hat{a}^+\hat{a} | \alpha \rangle - \langle \alpha | \alpha \rangle\right) =$$

$$= -\frac{1}{4}\left(\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1\right) =$$

$$= -\frac{1}{4}\left(\left(\alpha - \alpha^*\right)^2 - 1\right) = \left(Im\,\alpha\right)^2 + \frac{1}{4} \quad (11b)$$

$$\to \left(\Delta X_1\right)_\alpha^2 = \langle X_1^2 \rangle_\alpha - \langle X_1 \rangle_\alpha^2 = \langle \alpha | X_1^2 | \alpha \rangle - \langle \alpha | X_1 | \alpha \rangle^2 \overset{(9a,\,11a)}{=}$$

$$= \left(Re\,\alpha\right)^2 + \frac{1}{4} - \left(Re\,\alpha\right)^2 = \frac{1}{4} \quad (12a)$$

$$\to \left(\Delta X_2\right)_\alpha^2 = \langle X_2^2 \rangle_\alpha - \langle X_2 \rangle_\alpha^2 = \langle \alpha | X_2^2 | \alpha \rangle - \langle \alpha | X_2 | \alpha \rangle^2 \overset{(9b,\,11b)}{=}$$

$$= \left(Im\,\alpha\right)^2 + \frac{1}{4} - \left(Im\,\alpha\right)^2 = \frac{1}{4} \quad (12b)$$

Zatem $\boxed{\left(\Delta X_1 \Delta X_2\right)_\alpha = \frac{1}{4}}$ (13)

$$\to [\hat{X}_1, \hat{X}_2] \overset{(8)}{=} \frac{1}{4i}\left[\hat{a} + \hat{a}^+, \hat{a} - \hat{a}^+\right] = \frac{1}{4i}\left(\overset{0}{\cancel{[\hat{a}, \hat{a}]}} - \overset{1}{\cancel{[\hat{a}, \hat{a}^+]}} + \overset{-1}{\cancel{[\hat{a}^+, \hat{a}]}} - \overset{0}{\cancel{[\hat{a}^+, \hat{a}^+]}}\right)$$

$$= -\frac{1}{2i} = \frac{i}{2} \quad (14)$$

⚠ Dla każdego stanu $|\psi\rangle$ mamy, że $\left(\Delta X_1 \Delta X_2\right)_\psi \geq \frac{1}{4}$, zatem STAN KOHERENTNY MINIMALIZUJE ZASADĘ NIEOZNACZONOŚCI DLA OPERATORÓW KWADRATUR.