WSPÓŁRZĘDNE KRZYWOLINIOWE (q_1, q_2, q_3)

ukł. kartezjański:

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \qquad (0)$$

teraz:

$$q_1 = q_1(x,y,z) \qquad (1a)$$
$$q_2 = q_2(x,y,z) \qquad (1b)$$
$$q_3 = q_3(x,y,z) \qquad (1c)$$

Współczynniki Lamego:

$$\boxed{H_i = \left|\frac{\partial \vec{r}}{\partial q_i}\right|} \qquad (2) \quad , \; i \in \{1,2,3\}$$

Wersory:

$$\boxed{\hat{q}_i = \frac{1}{H_i}\frac{\partial \vec{r}}{\partial q_i}} \qquad (3)$$

$$\begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \end{bmatrix} = \underset{\text{macierz } 3\times 3}{A} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \quad (4a) \text{, zatem} \quad \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = A^{-1} \begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \end{bmatrix} \quad (4b)$$

$= A^T$, bo $\hat{q}_1, \hat{q}_2, \hat{q}_3$ są ortonormalne (*)

Dowolny wektor:

$$\vec{w} = w_1\hat{q}_1 + w_2\hat{q}_2 + w_3\hat{q}_3 = w_x\hat{i} + w_y\hat{j} + w_z\hat{z} \qquad (5)$$

Niech f → pewna funkcja skalarna

- gradient f, grad(f)

$$\boxed{\nabla f = \sum_{i=1}^{3} \frac{1}{H_i}\left(\frac{\partial f}{\partial q_i}\right)\hat{q}_i} \qquad (6)$$

- dywergencja $\vec{w}$, div$(\vec{w})$

$$\boxed{\nabla \cdot \vec{w} = \frac{1}{H_1H_2H_3}\left(\frac{\partial(w_1H_2H_3)}{\partial q_1} + \frac{\partial(H_1w_2H_3)}{\partial q_2} + \frac{\partial(H_1H_2w_3)}{\partial q_3}\right)} \qquad (7)$$

- rotacja $\vec{w}$, $\text{rot}(\vec{w})$

$$\nabla \times \vec{w} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} H_1 \hat{q}_1 & H_2 \hat{q}_2 & H_3 \hat{q}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ H_1 w_1 & H_2 w_2 & H_3 w_3 \end{vmatrix} \tag{8}$$

- laplasjan

→ skalarny: $\Delta f = \text{div}(\text{grad}(f))$

$$\Delta f = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2 H_3}{H_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1 H_3}{H_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{H_1 H_2}{H_3}\frac{\partial f}{\partial q_3}\right)\right] \tag{9a}$$

→ wektorowy: $\Delta \vec{w} = ?$

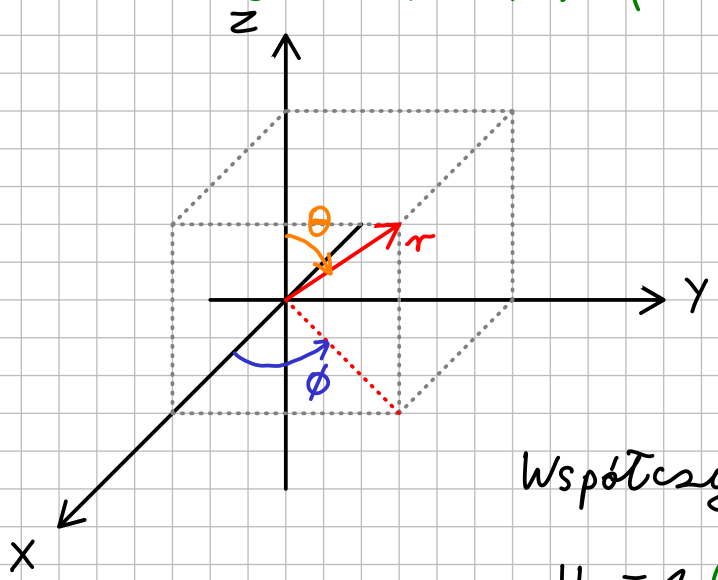
ukł. kartezjański $\quad \Delta \vec{w} = \hat{i}\Delta w_x + \hat{j}\Delta w_y + \hat{k}\Delta w_z \quad (9b')$

Jednak w ogólności (9b') NIE DZIAŁA dla współrzędnych krzywoliniowych ⚠

mamy w ogólności, że

$$\Delta \vec{w} = \underbrace{\nabla}_{\text{grad}}(\underbrace{\nabla \cdot}_{\text{div}} \vec{w}) - \underbrace{\nabla \times}_{\text{rot}}(\underbrace{\nabla \times}_{\text{rot}} \vec{w}) \tag{9b}$$

UKŁAD SFERYCZNY (MATEMATYCZNY)



$$\begin{cases} x = r\cos\phi\sin\theta & (1a) \\ y = r\sin\phi\sin\theta & (1b) \\ z = r\cos\theta & (1c) \end{cases}$$

Współczynniki Lamego:

$$H_r = 1 \quad (2a), \quad H_\theta = r \quad (2b), \quad H_\phi = r\sin\theta \quad (2c)$$

Wersory:

$$\begin{cases} \hat{r} = [\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta] & (3a) \\ \hat{\theta} = [\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta] & (3b) \\ \hat{\phi} = [-\sin\phi, \cos\phi, 0] & (3c) \end{cases}$$

zatem:

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \quad (4a)$$

$$\Big\downarrow \text{transpozycja}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \quad (4b)$$

stąd

$$\begin{cases} \hat{i} = \hat{r}\cos\phi\sin\theta + \hat{\theta}\cos\phi\cos\theta - \hat{\phi}\sin\phi & (3a') \\ \hat{j} = \hat{r}\sin\phi\sin\theta + \hat{\theta}\sin\phi\cos\theta + \hat{\phi}\cos\phi & (3b') \\ \hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta & (3c') \end{cases}$$

Niech teraz $\vec{w} = w_r\hat{r} + w_\theta\hat{\theta} + w_\phi\hat{\phi}$ (5)

- gradient f

$$\boxed{\nabla f \overset{(2),(6)}{=} \left(\frac{\partial f}{\partial r}\right)\hat{r} + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)\hat{\theta} + \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial \phi}\right)\hat{\phi}} \qquad (6)$$

- dywergencja $\vec{w}$

$$\operatorname{div}(\vec{w}) \overset{(2),(7)}{=} \frac{1}{r^2\sin\theta}\left(\frac{\partial(w_r r^2\sin\theta)}{\partial r} + \frac{\partial(w_\theta r\sin\theta)}{\partial\theta} + \frac{\partial(r w_\phi)}{\partial\phi}\right)$$

$$\boxed{\operatorname{div}(\vec{w}) = \frac{1}{r^2}\,\frac{\partial(r^2 w_r)}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial(w_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\,\frac{\partial w_\phi}{\partial\phi}} \qquad (7)$$

- rotacja

$$\operatorname{rot}(\vec{w}) \overset{(2),(8)}{=} \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r(\sin\theta)\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ w_r & r w_\theta & r(\sin\theta)\, w_\phi \end{vmatrix} =$$

$$= \frac{1}{r^2\sin\theta}\left(\hat{r}\,\frac{\partial(r(\sin\theta)w_\phi)}{\partial\theta} - \hat{r}\,\frac{\partial(r w_\theta)}{\partial\phi} - r\,\hat{\theta}\,\frac{\partial(r(\sin\theta)w_\phi)}{\partial r} + r\,\hat{\theta}\,\frac{\partial w_r}{\partial\phi} + r(\sin\theta)\,\hat{\phi}\,\frac{\partial(r w_\theta)}{\partial r} - r(\sin\theta)\,\hat{\phi}\,\frac{\partial w_r}{\partial\theta}\right) =$$

$$= \frac{\cancel{r}\,\hat{r}}{r^{\cancel{2}}\sin\theta}\left(\frac{\partial(w_\phi\sin\theta)}{\partial\theta} - \frac{\partial w_\theta}{\partial\phi}\right) - \frac{\cancel{r}\,\hat{\theta}}{r^{\cancel{2}}\sin\theta}\left(\sin\theta\,\frac{\partial(r w_\phi)}{\partial r} - \frac{\partial w_r}{\partial\phi}\right)$$

$$+ \frac{\cancel{r}\,\hat{\phi}\,\cancel{\sin\theta}}{r^2\,\cancel{\sin\theta}}\left(\frac{\partial(r w_\theta)}{\partial r} - \frac{\partial w_r}{\partial\theta}\right)$$

$$\boxed{\begin{aligned}\operatorname{rot}(\vec{w}) = {} & \frac{\hat{r}}{r\sin\theta}\left(\frac{\partial(w_\phi\sin\theta)}{\partial\theta} - \frac{\partial w_\theta}{\partial\phi}\right) + \frac{\hat{\theta}}{r}\left(\frac{1}{\sin\theta}\,\frac{\partial w_r}{\partial\phi} - \frac{\partial(r w_\phi)}{\partial r}\right) \\ & + \frac{\hat{\phi}}{r}\left(\frac{\partial(r w_\theta)}{\partial r} - \frac{\partial w_r}{\partial\theta}\right)\end{aligned}} \qquad (8)$$

- laplasjan (skalarny)

$$\Delta f \overset{(2),(9a)}{=} \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\right)\right]$$

$$\boxed{\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}} \qquad (9)$$

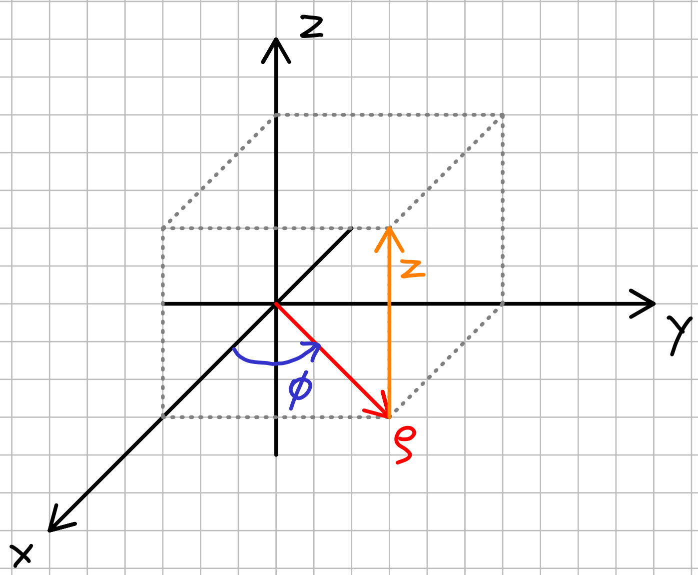
- iloczyn wektorowy :

$$\vec{A} = A_r\hat{r}+A_\theta\hat{\theta}+A_\phi\hat{\phi}$$
$$\vec{B} = B_r\hat{r}+B_\theta\hat{\theta}+B_\phi\hat{\phi}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$$

$$\boxed{\vec{A}\times\vec{B} = \hat{r}\left(A_\theta B_\phi - A_\phi B_\theta\right)+\hat{\theta}\left(A_\phi B_r - A_r B_\phi\right)+\hat{\phi}\left(A_r B_\theta - A_\theta B_r\right)} \qquad (10)$$

UKŁAD WALCOWY



$$\begin{cases} x = \varrho\cos\phi & (11a) \\ y = \varrho\sin\phi & (11b) \\ z = z & (11c) \end{cases}$$

Współczynniki Lamego

(proszę sobie przeliczyć z (2))

$$\begin{cases} H_\varrho = 1 & (12a) \\ H_\phi = \varrho & (12b) \\ H_z = 1 & (12c) \end{cases}$$

Wersory (por. (3))

$$\begin{cases} \hat{\varrho} = \hat{i}\cos\phi + \hat{j}\sin\phi & (13a) \\ \hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi & (13b) \\ \hat{z} = \hat{k} & (13c) \end{cases} \xrightarrow{(4a,b)} \begin{cases} \hat{i} = \hat{\varrho}\cos\phi - \hat{\phi}\sin\phi & (13a') \\ \hat{j} = \hat{\varrho}\sin\phi + \hat{\phi}\cos\phi & (13b') \\ \hat{k} = \hat{z} & (13c') \end{cases}$$

Niech

$$\vec{A} = A_\varrho\hat{\varrho}+A_\phi\hat{\phi}+A_z\hat{z} \qquad (14a)$$
$$\vec{B} = B_\varrho\hat{\varrho}+B_\phi\hat{\phi}+B_z\hat{z} \qquad (14b)$$

- gradient f

$$\boxed{\nabla f \overset{(6),(12)}{=} \left(\frac{\partial f}{\partial \varrho}\right)\hat{\varrho} + \frac{1}{\varrho}\left(\frac{\partial f}{\partial \phi}\right)\hat{\phi} + \left(\frac{\partial f}{\partial z}\right)\hat{z}} \tag{15}$$

- dywergencja $\vec{A}$

$$\boxed{\operatorname{div}(\vec{A}) \overset{(7),(12)}{=} \frac{1}{\varrho}\frac{\partial(\varrho A_\varrho)}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}} \tag{16}$$

- rotacja $\vec{A}$

$$\operatorname{rot}(\vec{A}) \overset{(8),(12)}{=} \frac{1}{\varrho}\begin{vmatrix} \hat{\varrho} & \varrho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \varrho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\varrho & \varrho A_\phi & A_z \end{vmatrix} =$$

$$= \frac{1}{\varrho}\left[\hat{\varrho}\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial(\varrho A_\phi)}{\partial z}\right) - \varrho\hat{\phi}\left(\frac{\partial A_z}{\partial \varrho} - \frac{\partial A_\varrho}{\partial z}\right) + \hat{z}\left(\frac{\partial(\varrho A_\phi)}{\partial \varrho} - \frac{\partial A_\varrho}{\partial \phi}\right)\right]$$

$$\boxed{\operatorname{rot}(\vec{A}) = \hat{\varrho}\left(\frac{1}{\varrho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\varrho}{\partial z} - \frac{\partial A_z}{\partial \varrho}\right) + \hat{z}\frac{1}{\varrho}\left(\frac{\partial(\varrho A_\phi)}{\partial \varrho} - \frac{\partial A_\varrho}{\partial \phi}\right)} \tag{17}$$

- laplasjan (skalarny)

$$\Delta f \overset{(9a),(12)}{=} \frac{1}{\varrho}\left[\frac{\partial}{\partial \varrho}\left(\varrho\frac{\partial f}{\partial \varrho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\varrho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\varrho\frac{\partial f}{\partial z}\right)\right]$$

$$\boxed{\Delta f = \frac{1}{\varrho}\frac{\partial f}{\partial \varrho} + \frac{\partial^2 f}{\partial \varrho^2} + \frac{1}{\varrho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}} \tag{18}$$

- iloczyn wektorowy:

$$\vec{A}\times\vec{B} \overset{(14)}{=} \begin{vmatrix} \hat{\varrho} & \hat{\phi} & \hat{z} \\ A_\varrho & A_\phi & A_z \\ B_\varrho & B_\phi & B_z \end{vmatrix}$$

$$\boxed{\vec{A}\times\vec{B} = \hat{\varrho}\left(A_\phi B_z - A_z B_\phi\right) + \hat{\phi}\left(A_z B_\varrho - A_\varrho B_z\right) + \hat{z}\left(A_\varrho B_\phi - A_\phi B_\varrho\right)} \tag{19}$$